King Fahd University of Petroleum and Minerals
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ICS 253 Discrete Structures I – Summer Term 2014 (2013-3) Final Exam - Time: 2 hours - Key Solution

Note: Full points will be awarded only if proper explanation (or reasoning) is stated.

1. Let R_n denote the number of strings of length n using the digits {0,1,2} that **do not contain** two consecutive 0s.

a) [3 pts.] Derive, using the appropriate diagram and explanation, the recurrence relation for R_n.

1/2	strings of length n-1 that *do not contain* two consecutive 0s; this is $2R_{n-1}$	
0	0	All these strings are excluded
0	1/2	strings of length n-2 that *do not contain* consecutive 1s; this is $2R_{n-2}$

$R_n = 2R_{n-1} + 2R_{n-2}$

b) [2 pts.] Compute R_1, R_2 and R_3 directly. Then use the recurrence found in (a) to compute R_3.

$R_1 = 3$ (all possible strings of length 1 are counted),
$R_2 = 8$ (from the 9 possible strings, we exclude "00"),
$R_3 = 22$ (all strings of length 3 (=27) except the strings: 000, 001, 002, 100, 200);
by Rec. , $R_3 = 2R_2 + 2R_1 = 2*8+2*3 = 22$.

2. [5 pts.] Given the linear nonhomogeneous recurrence relation $a_n = 6\, a_{n-1} - 9a_{n-2} + F(n)$.

a) Write the form for the homogenous solution (i.e., ignoring $F(n)$).

Characteristic Equation is $r^2 - 6r + 9 = (r-3)\,(r-3) = 0$; one repeated root, $r = 3$.
Thus, the homogeneous solution is $a_n = \mathrm{a}r^n + \mathrm{b}n\, r^n = \mathrm{a}3^n + \mathrm{b}n\, 3^n$ for some real number constants a, b.

b) Write the form for the particular solution if $F(n) = n\, 3^n$.

In general, for $F(n) = P(n)\, C^n$ where C is a root of multiplicity m,
the particular solution is of the form $a_n = n^m\, P(n)\, C^n$.
Because C=3 is a root with multiplicity m=2, the particular solution is $a_n = n^2\,(\mathrm{a} + \mathrm{b}n)\, 3^n$

c) Write the form for the particular solution if $F(n) = 3n^2$.

This is a case where the constant C in $F(n)$ is not a root. The form of the solution is $a_n = P(n)\, C^n$.
Because here C=1, the particular solution is $a_n = (\mathrm{a} + \mathrm{b}n + \mathrm{c}n^2)$ (1)

d) Write the equation that can be used to find the values of the constants in part (c).

We use Eq. (1) in part (c) and substitute the value for a_n in the original recurrence $a_n = 6a_{n-1} - 9a_{n-2} + 3n^2$. Thus, we get

$$(\mathrm{a} + \mathrm{b}n + \mathrm{c}n^2) = 6\,[\mathrm{a} + \mathrm{b}(n-1) + \mathrm{c}(n-1)^2\,] - 9\,[\mathrm{a} + \mathrm{b}(n-2) + \mathrm{c}(n-2)^2\,] + 3n^2$$

3. [4 pts.] In the questions below write (*include proper reasoning*) a simple formula for the generating function associated with the given sequence.

a) 9, 1, 1, 1, 9, 1, 1, 1, 9, ….

The sequence 1,0,0,0,1,0,0,0,1, … corresponds to $1/(1-x^4)$
Thus, the sequence 8,0,0,0,8,0,0,0,8, … corresponds to $8/(1-x^4)$
By adding the sequence 1,1,1, …, we get the sequence in question
Thus, the sequence in question has its GF as $8/(1-x^4) + 1/(1-x)$

b) 4, 7, 10, 13, …

The sequence is a sum of two sequences: 1,1,1, … and 3,6,9,12, [this is 3 * (1,2,3, ..)]
Thus, the GF for the given sequence is $1/(1-x) + 3/(1-x)^2$
$= (1 - x + 3) / (1-x)^2 = (4 - x)/ (1-x)^2$

4. [4 pts.] Find, with proper explanation, the coefficient of x^8 in the power series of these functions:
a) $(1+x^2 + x^4 + x^8)^5$

We need to select five terms that (when multiplied) add up to x^8
This is achieved as follows:
Selecting one x^8 from one of the parts (and x^0 from other parts) $\Rightarrow C(5,1)=5$
OR two x^4 $\Rightarrow C(5,2)=10$
Or one x^4 and two x^2 $\Rightarrow C(5,1)*C(4,2)= 30$
Or four x^2 $\Rightarrow C(5,4)=5$
Thus, answer = 5+10+30+5 = 50

b) $[1/ (1+x)] [10x + 9x^2 + 8x^3 + 7x^4]$. Hint: First, express the first bracket in expanded form.

This is a product of two GFs

$(1-x + x^2 - x^3 + \dots) (0+10x + 9x^2 + 8x^3 + 7x^4)$

View the coefficients of the first GF as a_i's and the second GF as b_j's.

Then the coefficients of x^8 in the product is given as $a_0b_8 + a_1b_7 + \dots + a_8b_0$
$= a_4b_4 + a_5b_3 + a_6b_2 + a_7b_1 = 7 - 8 + 9 - 10 = -2$

5. [6 pts.] How many ways are there to select 10 ice cream cones when there are 12 flavors such that:
[(a) At least one cone of every flavor is chosen] → At least one cone of flavor 1 or flavor 2.
A [B] : ways with at least one cone of flavor 1[2]; $|A\cup B| = |A| + |B| - |A\cap B|$;
$|A|$ (reserve one cone from flavors 1 and choose rem. 9 cones) = $C(9+12-1,9) = C(20,9)$
$|A\cap B|$ (reserve two cones from flavors 1, 2 and choose rem. 8 cones) = $C(8+12-1,8) = C(19,8)$
$\Rightarrow$ *Answer* = $C(20,9) + C(20,9) - C(19,8)$

(b) At most one cone of every flavor (i.e. a flavor is either chosen or not chosen)

Selection (without repetition) of 10 flavors from 12 flavors
$\Rightarrow$ *Answer* = $C(12, 10) = C(12, 2) = 12*11/2=66$

(c) Exactly two flavors

Choose two flavors (12-choose-2) multiplied by the number of ways of buying 10 cones from 2 flavors
$\Rightarrow$ *Answer* = $C(12, 2) * C(10+2-1, 10) = (12*11)/2 * C(11, 10) = 66 * 11 = 726$

6. [6 pts.] Compute a non-recursive formula (in terms of $\boldsymbol{k}$) for $\boldsymbol{a_k}$ (the ***coefficient of*** $\boldsymbol{x^k}$) in the sequence determined by the given generating function. Also give the values of the ***first three terms*** of the sequence.

a) $(2+x)^5$

By Binomial Theorem, the k-th term is $a_k = C(5,k)\, 2^{5-k} x^k$ for $k = 0$ to 5; $a_k = 0$ for $k > 5$.
Thus, $a_0 = C(5,0)\, 2^5 = 32$; $a_1 = C(5,1)\, 2^4 = 5*16 = 80$; $a_2 = C(5,2)\, 2^3 = 80$

b) $x^2 / (1 - cx)$ for some fixed real number c.

This corresponds to the sequence (replace x in GP by cx): $x^2\ [1+ (cx) + (cx)^2 + (cx)^3 + \ldots]$
We can easily see that $a_0 = a_1 = 0$ and $a_2 = 1$
In general, for $k \geq 2$, $a_k = (c)^{k-2}$

c) $1 / (1+3x^2)^{10}$ **[ICS Fall 2014 Exam: *Skip this question*]**

By the Ext. Bin. Theorem, **the k-th term in the expansion of $1 / (1+3x^2)^{10}$ is $C(-10, k)\,(3x^2)^k$**

Thus, $\boldsymbol{a_k}$ **(*Coefficient of* x^k) = $C(-10, k/2)$** $(3)^{k/2}$

$a_0 = C(-10,\ 0)\,(3)^0 = 1$
$a_1 = C(-10,\ 1/2)\,(3)^{1/2} = 0$
$a_2 = C(-10,\ 1)\,(3)^1 = -30$
Note: $1 / (1+3x^2)^{10}$ $= [$**$1+(-3x^2) + (-3x^2)^2$** $+\ldots]^{\mathbf{10}}$
From this, we see that $a_1 = 0$ (no way to get x) and $a_2 =$ (10 ways of getting $\boldsymbol{x^2}$, each contributes -3) $= -30$

7. [6 pts.] Consider the recurrence $a_n = 3a_{n-1} + 2^n$ with $a_0 = 2$. Solve this recurrence as specified below.

a) Using backward substitution.

$a_n = 3\, a_{n-1} + 2^n$
$= 3\,(3a_{n-2} + 2^{n-1}) + 2^n$
$= 3^2\, a_{n-2} + 3.2^{n-1} + 2^n$
$= 3^i\, a_{n-i} + 3^{i-1}\, 2^{n-i+1} + \ldots + 3.\, 2^{n-1} + 2^n$ (to reach a_0, we let $i = n$)
$= 3^n\, a_0 + 3^{n-1}\, 2 + \ldots + 3.\, 2^{n-1} + 3^0.\, 2^n$
$= 3^n\, a_0 + 2^n\ [(3/2)^{n-1} + \ldots + (3/2)^1 + (3/2)^0]$

[We use Geometric Progression, $1+a+a^2+ \ldots + a^k = (a^{k+1} - 1)/(a - 1)$]

$= 3^n\, a_0 + 2^n\ [((3/2)^n - 1)/((3/2) - 1)]$
$= 3^n * 2 + 2^n\ [((3/2)^n - 1)/((3/2) - 1)]$! verify $a_1 = 8$, (by rec $a_1 = 3 * 2 + 2 = 8$)

b) Using the appropriate formula given by the LNHR method.

Ch. Eq. : $r - 3 = 0$; one root $r = 3$. Gen. Solution is *homogenous solution* + *particular solution*
Homogenous Solution is $a_n = \text{a} . 3^n$
Particular solution: form of $F(n) = p(n) * C^n$
In this case, $C = 2$ which is not a root of the homogenous part. Thus, for the particular solution $a_n = \text{b}\, 2^n$
To find b we use the eq. $a_n = 3\, a_{n-1} + 2^n$ and by sub. $a_n = \text{b}\, 2^n$
$\Rightarrow$ $\text{b}\, 2^n = 3\, [\text{b}\, 2^{n-1}] + 2^n$ $\Rightarrow$ $\text{b} = 3/2\ \text{b} + 1$ $\Rightarrow$ $\text{b} = -2$
Thus, the gen. solution is $a_n = \text{a} . 3^n - 2.\, 2^n = \text{a} . 3^n - 2^{n+1}$
We solve for a, $a_0 = 2 = \text{a} . 3^0 - 2 \Rightarrow \text{a} = 4$
$\Rightarrow$ **gen. solution** is $\boldsymbol{a_n = 4 . 3^n - 2^{n+1}}$.
Verify: By the rec. $a_1 = 8$, $a_2 = 28$; By the solution formula: $a_1 = 4 . 3 - 2^2 = 8$; $a_2 = 4 . 9 - 2^3 = 28$.

8. [6 pts.] Fill in the blanks.

a. If a biased coin with *Prob*(Head) = 1/3 is flipped 12 times, then the expected number of Tails is **8** .

b. The coefficient of x^k in $g(x) = 1/(1-x^2)^{10}$ (simple form only) gives the number of ways of buying ***k*** cones of ice cream from **10** flavors where *the number of cones selected from any flavor is even*.

c. The number of ways to select ***r*** objects of ***n*** different kinds if we must select one object of each kind is given by the generating function $(x + x^2 + x^3 + \ldots)^n$ which can be simplified as $x^n / (1-x)$.

d. The probability of having at least six heads in 7 coin flips is **1/16** . *Show the steps for this.*

Answer = Prob. of 7 Heads or 6 Heads = $1/2^7 + C(7,6)/2^7$ = 8/128= 1/16.

9. [8 pts.] A CS class has three sections *A*, *B*, *C* with 7, 9 and 11 students, respectively.
A committee of 4 students is to be selected at random (without replacement) from this class.

a) How many different committees are there with exactly one student from section *A*?

Choose one student from section *A* then choose three students from other sections: *C*(7,1)**C*(20,3)

Note: Deal with part (b) [also (c) and (d)] as consisting of cases. ***Skip computation of final numbers.***

b) How many committees are there if a committee is to include at least one student from each section?
Hint: *at least one* means *one, two, …, etc*.

As cases: 2,1,1 or 1,2,1 or 1,1,2
This gives *C*(7,2)**C*(9,1)**C*(11,1) + *C*(7,1)**C*(9,2)**C*(11,1) + *C*(7,1)**C*(9,1)**C*(11,2)
= 21*9*11 +7*36*11+ 7*9+55 = 8136

c) Compute the probability that all members of the committee come from the same section.

Answer = [*C*(7,4) + *C*(9,4) + *C*(11,4)] / *C*(27,4)

d) Let ***X*** be a random variable defined as the *number of students from section* ***A*** in the randomly selected committee. Compute **1)** ***the probability distribution for X*** and **2)** ***the expected value for X***.

$P(X=i) = C(7,i)*C(20,4-i)/ C(27,4)$ for i=0,1,2,3,4
$Exp(X)$ = sum (for i=0 to 4) of $P(X=i)*i$